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AFFINE TRANSFORMS OF THREE-DIMENSIONAL ANISOTROPIC MEDIA AND EXPLICIT FORMULAS FOR FUNDAMENTAL MATRICES

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The concept of classes of algebraically equivalent anisotropic three-dimensional media is introduced: elastic fields in such media are related by simple algebraic expressions. An explicit formula is obtained for a fundamental matrix with ten free constants in the elastic modulus tensor (rather than for five free constants, as in the famous case of transverse isotropy). A hypothesis is formulated and several questions are posed, which are related to the notion of algebraic equivalence under discussion.

Key words: *affine transform, three-dimensional system of equations of the elasticity theory, fundamental matrix.*

1. Affine Transforms in the Theory of Elasticity. Kröner [1] constructed a fundamental matrix F (Kelvin tensor) for a transversely isotropic homogeneous medium, characterized by five free constants in the elastic modulus tensor (or a family with allowance for rotation of the Cartesian coordinate system). For 50 years, the number of free constants in the explicit formula for F has not been increased, despite the attempts made to develop the method suggested in [1] or to use other approaches (see [2, 3] and other references). Certainly, the transform of the matrix F by means of the direct and inverse Fourier transforms is available, but the thus-obtained formula cannot be called explicit; moreover, its application in computational schemes (e.g., for deriving and solving the boundary integral equations) is rather problematic because it is difficult to reach acceptable accuracy of numerical conversion of the Fourier transform. In the present work, in particular, we obtained a formula for the fundamental matrix, which admits

$$10 = 5 + (9 - 1 - 3) \tag{1.1}$$

free constants in the elastic modulus tensor. Here 5 is the number of constants in the case of a transversely isotropic material and 9 is the number of elements of a nondegenerate 3×3 matrix

$$m = (m_{jk})_{j,k=1,2,3}; \tag{1.2}$$

the subtracted 1 takes into account the normalization condition

$$\det m = 1; \tag{1.3}$$

the subtracted 3 is caused by possible rotations of the Cartesian coordinate system. In other words, with the fundamental matrix $F(x)$ being available, we use affine transforms of the coordinates

$$x \mapsto \boldsymbol{x} = mx \tag{1.4}$$

to construct a new matrix function

$$\boldsymbol{F}(\boldsymbol{x}) = (m^{\mathrm{t}})^{-1} F(m^{-1}\boldsymbol{x}) m^{-1} \tag{1.5}$$

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and prove that it is fundamental for a certain new elastic anisotropic medium. The elastic modulus tensor for the latter is found by simple algebraic operations over the initial tensor (see Sec. 2). The bold quantities in Eqs. (1.4), (1.5) and in what follows are the quantities in transformed coordinates.

The use of affine transforms in the elasticity theory is based on two observations. First, all handbooks on the theory of differential equations in partial derivatives begin from a description of the procedure of reduction of second-order equations to a canonical form. In particular, the replacement of coordinates (1.4) transforms the scalar differential operator $\nabla_x^{\mathrm{t}} a \nabla_x$ into the Laplacian $\Delta_x = \nabla_x^{\mathrm{t}} \nabla_x$, where ∇_x is the column $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^{\mathrm{t}}$; m is the root of a symmetrical, positively determined numerical matrix a^{-1} of size 3×3. Second, the issue of reduction of the matrix operator of the system of equations of the elasticity theory to a canonical form becomes fairly logical after passing in a fixed system of Cartesian coordinates to columns of displacements $u = (u_1, u_2, u_3)^{\mathrm{t}}$ and also to columns of strains and stresses of height 6:

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \alpha^{-1}\varepsilon_{23}, \alpha^{-1}\varepsilon_{31}, \alpha^{-1}\varepsilon_{12})^{\mathrm{t}}; \tag{1.6}$$

$$\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \alpha^{-1}\sigma_{23}, \alpha^{-1}\sigma_{31}, \alpha^{-1}\sigma_{12})^{\mathrm{t}}. \tag{1.7}$$

A similar form of the equation was proposed in the 1930s by Lekhnitskii (see, e.g., [4]) and was later improved by Nazarov [5]. Thus, the terms $\alpha^{-1} = \sqrt{2}$ added to Eq. (1.6) balance the natural norms of the column and the strain tensor [this also refers to stresses (1.7)]. Hence, the replacement of coordinates (1.4) with an orthogonal matrix (1.2) corresponds to orthogonal transforms of the columns (1.6) and (1.7) (see [5, § 2.1] and Sec. 3). The relation between the columns of displacements and strains

$$\varepsilon = D(\nabla_x)u$$

is now implemented with the use of a 6×3 matrix $D(\nabla_x)$ of differential operators of the first order:

$$D(\nabla_x)^{\mathrm{t}} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \alpha\partial_3 & \alpha\partial_2 \\ 0 & \partial_2 & 0 & \alpha\partial_3 & 0 & \alpha\partial_1 \\ 0 & 0 & \partial_3 & \alpha\partial_2 & \alpha\partial_1 & 0 \end{pmatrix}, \qquad \partial_j = \frac{\partial}{\partial x_j}, \qquad \alpha = \frac{1}{\sqrt{2}}. \tag{1.8}$$

Hooke's law acquires the form

$$\sigma = A\varepsilon, \tag{1.9}$$

where A is a symmetrical, positively determined matrix of size 6×6 whose elements are proportional to components of the elastic modulus tensor (see [5, § 2.1] for more details). It is manipulations with the matrices A and $D(\nabla_x)$ in the affine transform (1.4) that allow us to obtain formula (1.5) for the fundamental matrix (see Sec. 4). In Sec. 3, in addition, we introduce and discuss the concept of algebraically equivalent media for which the stress–strain state is recalculated by means of multiplication by appropriate matrices.

For a plane problem of the elasticity theory, these transforms were used in [6–8] for different purposes. Certainly, fundamental matrices were out of the question in these papers, as these matrices are known for any anisotropy of a plane medium (see [9] and other papers). At the same time, the classification of systems of equations of the two-dimensional elasticity theory was completed: it was ultimately established in [7] that an arbitrary anisotropic two-dimensional medium is algebraically equivalent to an orthotropic medium with a fourth-order axis of symmetry and an additional relation $A_{11} \geq A_{12} + A_{33}$. Such a conclusion could not be drawn for spatial media, and we only put forward a hypothesis about the simplest representatives of all anisotropic three-dimensional media in Sec. 4.

2. Matrix Recording of the Problem of the Theory of Elasticity and Transforms. Let $G \subset \mathbb{R}^3$ be a body with a piecewise-smooth boundary ∂G affected by mass forces f. We set the force g on the portion T of the surface ∂G and the external displacements h on the surface portion $S = \partial G \setminus \overline{T}$. One of the sets S or T can be empty, and both sets are empty in the case $G = \mathbb{R}^3$. Direct computations (see [5, § 2.1]) show that the corresponding problem of the elasticity theory is written in the notation introduced in (1.6)–(1.9) as follows:

$$D(-\nabla_x)^{\mathrm{t}} A D(\nabla_x) u(x) = f(x), \qquad x \in G; \tag{2.1}$$

$$D(n(x))^{\mathrm{t}} A D(\nabla_x) u(x) = g(x), \quad x \in T, \qquad u(x) = h(x), \quad x \in S. \tag{2.2}$$

Here $n(x)$ is the unit vector of the external normal at the point $x \in \partial G$.

After replacement of the coordinates (1.4), the sets G, T, and S are transformed to the sets $\boldsymbol{G}$, $\boldsymbol{T}$, and $\boldsymbol{S}$, respectively; for example, $\boldsymbol{G} = \{\boldsymbol{x}\colon\ m^{-1}\boldsymbol{x} \in G\}$. The unit vector (column) $\boldsymbol{n}$ of the normal to the surface $\partial\boldsymbol{G}$ is calculated as

$$\boldsymbol{n}(\boldsymbol{x}) = |(m^{\mathrm{t}})^{-1}n(m^{-1}\boldsymbol{x})|^{-1}(m^{\mathrm{t}})^{-1}n(m^{-1}\boldsymbol{x}), \tag{2.3}$$

where $|v| = (v_1^2 + v_2^2 + v_3^2)^{1/2}$ is the length of the column $v \in \mathbb{R}^3$.

In the domain $\boldsymbol{G}$, we introduce four columns and a symmetrical, positively determined matrix

$$\begin{gathered}\boldsymbol{u}(\boldsymbol{x}) = (m^{\mathrm{t}})^{-1}u(m^{-1}\boldsymbol{x}), \quad \boldsymbol{f}(\boldsymbol{x}) = mf(m^{-1}\boldsymbol{x}), \quad \boldsymbol{h}(\boldsymbol{x}) = (m^{\mathrm{t}})^{-1}h(m^{-1}\boldsymbol{x}),\\ \boldsymbol{g}(\boldsymbol{x}) = |(m^{\mathrm{t}})^{-1}n(m^{-1}\boldsymbol{x})|^{-1}mg(m^{-1}\boldsymbol{x}), \qquad \boldsymbol{A} = MAM^{\mathrm{t}},\end{gathered} \tag{2.4}$$

where the auxiliary matrix M of size 6×6 is constructed from the elements of the matrix (1.2) and has the form

$$\begin{pmatrix} m_{11}^2 & m_{12}^2 & m_{13}^2 & \sqrt{2}\,m_{12}m_{13} & \sqrt{2}\,m_{11}m_{13} & \sqrt{2}\,m_{11}m_{12} \\ m_{21}^2 & m_{22}^2 & m_{23}^2 & \sqrt{2}\,m_{22}m_{23} & \sqrt{2}\,m_{21}m_{23} & \sqrt{2}\,m_{21}m_{22} \\ m_{31}^2 & m_{32}^2 & m_{33}^2 & \sqrt{2}\,m_{32}m_{33} & \sqrt{2}\,m_{31}m_{33} & \sqrt{2}\,m_{31}m_{32} \\ \sqrt{2}\,m_{21}m_{31} & \sqrt{2}\,m_{22}m_{32} & \sqrt{2}\,m_{23}m_{33} & m_{23}m_{32} + m_{22}m_{33} & m_{23}m_{31} + m_{21}m_{33} & m_{22}m_{31} + m_{21}m_{32} \\ \sqrt{2}\,m_{11}m_{31} & \sqrt{2}\,m_{12}m_{32} & \sqrt{2}\,m_{13}m_{33} & m_{13}m_{32} + m_{12}m_{33} & m_{13}m_{31} + m_{11}m_{33} & m_{12}m_{31} + m_{11}m_{32} \\ \sqrt{2}\,m_{11}m_{21} & \sqrt{2}\,m_{12}m_{22} & \sqrt{2}\,m_{13}m_{23} & m_{13}m_{22} + m_{12}m_{23} & m_{13}m_{21} + m_{11}m_{23} & m_{12}m_{21} + m_{11}m_{22} \end{pmatrix}.$$

This object appears owing to replacement of variables in the differential operator (1.8), namely,

$$D(\nabla_x) = M^{\mathrm{t}}D(\nabla_{\boldsymbol{x}})(m^{\mathrm{t}})^{-1}. \tag{2.5}$$

A similar matrix (without the factors $\sqrt{2}$) was given in [4, § 1.5] in discussing the recalculation of the components of the strain and stress tensors owing to rotation of the axes of Cartesian coordinates; arbitrary affine transforms, however, were not analyzed in [4].

According to relations (2.3)–(2.5), the mixed boundary-value problem (2.1), (2.2) in the new coordinates $\boldsymbol{x}$ looks as follows:

$$D(-\nabla_{\boldsymbol{x}})^{\mathrm{t}}\boldsymbol{A}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \boldsymbol{G}; \tag{2.6}$$

$$D(\boldsymbol{n}(\boldsymbol{x}))^{\mathrm{t}}\boldsymbol{A}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{T}, \qquad \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{S}. \tag{2.7}$$

The key issue of the present work is retaining the structure of the matrix differential operators in problem (2.6), (2.7).

The solution of problem (2.6), (2.7) being known for a material with an elastic modulus matrix $\boldsymbol{A}$, the elastic fields in problem (2.1), (2.2) for a materials with a matrix A are reconstructed by the formulas

$$\begin{gathered}u(x) = m^{\mathrm{t}}\boldsymbol{u}(mx),\\ \varepsilon(x) = M^{\mathrm{t}}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x})|_{\boldsymbol{x}=mx}, \qquad \sigma(x) = M^{-1}\boldsymbol{A}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x})|_{\boldsymbol{x}=mx}.\end{gathered} \tag{2.8}$$

It should be emphasized that system (2.6) and the boundary conditions (2.7) do not result from rewriting the problem of the elasticity theory (2.1), (2.2) with the use of physical components of the displacement vector and strain and stress tensors in a new non-orthogonal coordinate system $\boldsymbol{x}$. In such a transform, the structure of differential operators changes, and they are no longer expressed via the matrix $D(\nabla_{\boldsymbol{x}})$ (except for the case of an orthogonal matrix m, which describes rotation of the Cartesian coordinate system [4]). In other words, the columns $\boldsymbol{u}(\boldsymbol{x})$ and $D(\nabla_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{A}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}(\boldsymbol{x})$ cannot be interpreted as physical fields that refer to the displacement vector and the strain and stress tensors; they should be considered as auxiliary objects that allow calculation of real elastic fields.

The boundary-value problem (2.6), (2.7) is still a problem of the elasticity theory, but the elastic material is described by the matrix $\boldsymbol{A}$ from Eq. (2.3). Relation (2.8) between the solutions of problem (2.1), (2.2) and problem (2.6), (2.7) allows us to speak about the algebraic equivalence of elastic media defined by the matrices A and $\boldsymbol{A}$. The conclusions that follow from this observation are discussed below.

It is of interest that, together with the system of equations, the boundary conditions (2.2) in stresses and displacements are also "correctly" transformed. The third boundary condition (the normal displacement and shear stresses being set on the surface) is distorted under these changes, because the affine transform in the general case changes the angles. The conditions of conjugation in the problem of deformation of a piecewise-homogeneous body $G = G^+ \cup G^-$ are inherited in passing to the body $\boldsymbol{G} = \boldsymbol{G}^+ \cup \boldsymbol{G}^-$ only if the replacement of coordinates (1.4) is common for both parts of the composite body. The approach proposed is inapplicable for spectral and, hence, for dynamic problems, as substitutions (1.4) and (2.8) bring the system

$$D(\nabla_x)^{\mathrm{t}} A D(\nabla_x) u(x) = \lambda \rho u(x) \qquad (x \in G)$$

to the form

$$D(\nabla_{\boldsymbol{x}})^{\mathrm{t}} \boldsymbol{A} D(\nabla_{\boldsymbol{x}}) \boldsymbol{u}(\boldsymbol{x}) = m m^{\mathrm{t}} \lambda \rho \boldsymbol{u}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \boldsymbol{G})$$

though it contains the matrix mm^{t}, which turns to a unit matrix only in the case of an orthogonal matrix m.

3. Classes of Algebraically Equivalent Elastic Media. Calculating the determinant of the matrix M, we find that, owing to condition (1.3),

$$\det M = (\det m)^4 = 1. \tag{3.1}$$

Multiplication of the matrices M^1 and M^2 corresponding to two affine transforms m^1 and m^2 establishes the property

$$m = m^1 m^2 \qquad \Rightarrow \qquad M = M^1 M^2. \tag{3.2}$$

As $M = \mathbb{I}_6$ for $m = \mathbb{I}_3$ ($\mathbb{I}_n$ is the unit matrix of size $n \times n$), statement (3.2) guarantees that

$$m \mapsto M \quad \Rightarrow \quad m^{-1} \mapsto M^{-1}. \tag{3.3}$$

We should note another obvious fact:

$$m \mapsto M \quad \Rightarrow \quad m^{\mathrm{t}} \mapsto M^{\mathrm{t}}.$$

From here and from relation (3.2), it follows that the orthogonal 6×6 matrix M corresponds to the orthogonal 3×3 matrix m. The latter statement is invalid without introducing the terms α and α^{-1} into definitions (1.6), (1.7), and (1.8).

According to assumption (1.3) and properties (3.1)–(3.3), the mapping $m \mapsto M$ realizes the group $\mathfrak{N}_3$ as a subgroup $\mathfrak{M}$ of the group $\mathfrak{N}_6$; $\mathfrak{N}_n$ is a group (with respect to the product) of $n \times n$ matrices with a unit determinant. Symmetrical matrices of size 6×6 form a lineal $\mathfrak{S}$ (with respect to addition and multiplication by scalars), and positively determined matrices form a convex open cone $\mathfrak{S}_+$ in this lineal. For a fixed matrix $M \in \mathfrak{N}_6$, the mapping $A \mapsto M^{\mathrm{t}} A M$ is an isomorphism in $\mathfrak{S}$, which retains positive determinacy. Thus, owing to properties (3.1)–(3.3), the subgroup $\mathfrak{M}$ for each element $A \in \mathfrak{S}_+$ determines the class $\mathfrak{m}(A) \subset \mathfrak{S}_+$ of equivalent elements, i.e., for each $\boldsymbol{A} \in \mathfrak{m}(A)$, there is an element $M \in \mathfrak{M}$ such that $\boldsymbol{A} = M^{\mathrm{t}} A M$. Media with matrices of elastic moduli A^1 and A^2 are called algebraically equivalent if $A^1 \in \mathfrak{m}(A^2)$ or, which is equivalent by virtue of (3.3), $A^2 \in \mathfrak{m}(A^1)$.

4. Examples and Hypothesis. The introduced relation of equivalence is naturally transferred to the sets of matrices from $\mathfrak{S}_+$. Let $\mathfrak{L}_+$ and $\mathfrak{K}_+$ be families of matrices that describe transversely isotropic and orthotropic media, respectively. Instead of stiffness matrices entering into these classes, it is more convenient to operate with compliance matrices L and K (inverse matrices to stiffness matrices), which have the following form in accordance with [4]:

$$L = \begin{pmatrix} l_{11} & l_{12} & l_{13} & 0 & 0 & 0 \\ l_{12} & l_{11} & l_{13} & 0 & 0 & 0 \\ l_{13} & l_{13} & l_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & l_{66} \end{pmatrix}, \qquad K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 0 & 0 \\ k_{31} & k_{32} & k_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{66} \end{pmatrix}. \tag{4.1}$$

In these matrices,

$$l_{66} = 2(l_{11} - l_{12}), \tag{4.2}$$

and the elements k_{pq} are expressed in terms of the technical constants as

$$\begin{gathered} k_{pp} = E_p^{-1}, \qquad k_{3+p,3+p} = G_p^{-1}, \qquad p = 1,2,3, \\ k_{pq} = -\nu_{pq} E_p^{-1}, \qquad p,q = 1,2,3, \ p \neq q. \end{gathered} \tag{4.3}$$

In Eq. (4.3), G_p are the shear moduli and E_p and ν_{pq} are Young's moduli and Poisson's ratios, which obey three conditions:

$$\nu_{21}E_1 = \nu_{12}E_2, \qquad \nu_{13}E_3 = \nu_{31}E_1, \qquad \nu_{32}E_2 = \nu_{23}E_3.$$

We find a family of orthotropic media, which are related to transversely isotropic media by the affine transform (1.4) with a diagonal matrix

$$m^\mu = \mathrm{diag}\,\{\mu, \mu^{-1}, 1\}. \tag{4.4}$$

By virtue of the last formula in (2.4), the compliance matrix $L \in \mathfrak{L}_+$ in (4.1) after the affine transform acquires the form $M^{-1}LM^{-1}$, where M is a diagonal matrix

$$M = \mathrm{diag}\,\{\mu^2, \mu^{-2}, 1, \mu^{-1}, \mu, 1\}.$$

Hence, the equality $K = M^{-1}LM^{-1}$ is satisfied if

$$k_{11} = \mu^{-4}l_{11},\ k_{22} = \mu^4 l_{11}, \quad k_{13} = \mu^{-2}l_{13},\ k_{23} = \mu^2 l_{13}, \quad k_{44} = \mu^2 l_{44},\ k_{55} = \mu^{-2}l_{44}; \tag{4.5}$$

$$k_{12} = l_{12}, \quad k_{33} = l_{33}, \quad k_{66} = l_{66} = 2(l_{11} - l_{12}). \tag{4.6}$$

By virtue of relations (4.3), Eqs. (4.5) are soluble with respect to l_{11}, l_{13}, and l_{44} if and only if the quantity μ^4 coincides with the following relations equal to each other:

$$\frac{\nu_{13}}{\nu_{23}} = \frac{G_2}{G_1} = \sqrt{\frac{E_1}{E_2}}.$$

In solving Eqs. (4.6), there is one more constraint

$$2\sqrt{E_1E_2} = E_1E_2G_3^{-1} - \nu_{12}E_2 - \nu_{21}E_1.$$

The resultant orthotropic medium possesses six $(9 - 3 = 5 + 1)$ free constants, but the fundamental matrix for this medium is determined by the formula derived in [1] and modified in accordance with definitions (1.5) and (4.4). The number of free constants cannot be increased by one with the use of the diagonal matrix m because the affine transform (1.4) with $n^\rho = \mathrm{diag}\,\{\rho, \rho, \rho^{-2}\}$ converts the compliance matrix L from formula (4.1) to the matrix

$$\boldsymbol{L} = \begin{pmatrix} l_{11}\rho^{-4} & l_{12}\rho^{-4} & l_{13}\rho^2 & 0 & 0 & 0 \\ l_{12}\rho^{-4} & l_{11}\rho^{-4} & l_{13}\rho^2 & 0 & 0 & 0 \\ l_{13}\rho^2 & l_{13}\rho^2 & l_{33}\rho^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{44}\rho^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{44}\rho^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_{66}\rho^{-4} \end{pmatrix}. \tag{4.7}$$

All diagonal 3×3 matrices with a unit determinant can be represented in the form $m^\mu n^\rho$, and the elements in (4.7) are still related as (4.2). By properly choosing the parameter ρ, we can obtain a transversely isotropic medium with matrix (4.7) with three identical Young's moduli or three identical shear moduli.

Let us return to considering media without properties of symmetry. The equality

$$21 - 3 = 13 + (9 - 1 - 3) \tag{4.8}$$

allows us to put forward the following hypothesis:

$$\mathfrak{S}_+ = \mathfrak{m}(\mathfrak{P}_+) \tag{4.9}$$

($\mathfrak{P}_+$ is a class of matrices of elastic moduli for monoclinic media). In other words, for each matrix A, there is a matrix m and an affine transform (1.4), which confer a plane Π of elastic symmetry to a medium with a matrix $\boldsymbol{A}$. Thus, instead of solving problem (2.1), (2.2) for an arbitrary anisotropic body G, it is sufficient to solve problem (2.6), (2.7) for a material possessing the above-mentioned additional symmetry. The position of the plane Π is determined by the matrix A but not by the body geometry G; yet, if the lucky chance is that $\boldsymbol{G}$ is a cylinder with the centerline perpendicular to the plane Π, we can continue the simplifying algebraic transforms and reduce an antiplane problem to an isotropic one and a plane problem to an orthotropic one with a fourth-order axis of symmetry (see [6, 7]).

In Eq. (4.8), 21 is the number of elements of the symmetrical 6×6 matrix located above the main diagonal, 9, 3, and 1 have the same meaning as in Eq. (1.1), and 13 is the number of free constants in the matrix A for a medium with a plane of elastic symmetry.

In addition to simple counting (4.8) of free constants, we are not aware of any arguments supporting the validity of hypothesis (4.9). Note, in a two-dimensional situation, we have $6 - 1 = 3 + (4 - 1 - 1)$, where 6 is the total number of free constants; the term 3 takes into account Young's modulus $E_1 = E_2$, Poisson's ratio $\nu_{12} = \nu_{21}$, and shear modulus G for a specific orthotropic medium; the subtracted 1 corresponds to rotation in the plane, and the normalization condition (1.3) stands at the very end of the formula (see [6, 7]).

5. Some Invariants of Affine Transforms of Elastic Media. As the elements of volume and surface area before and after the affine transform are related as

$$dx = |\det m|^{-1} d\boldsymbol{x} = d\boldsymbol{x}, \qquad ds_x = |(m^{\mathrm{t}})^{-1} n(m^{-1}\boldsymbol{x})|\, ds_{\boldsymbol{x}},$$

the conversion formulas (2.4) establish the coincidence of the functionals of potential strain energy (elastic energy minus the work of external forces) in problem (2.1), (2.2) and problem (2.6), (2.7):

$$(1/2)(AD(\nabla_x)u, D(\nabla_x)u)_G - (u, f)_G - (u, g)_S + (AD(\nabla_x)u, h)_T$$

$$= (1/2)(\boldsymbol{A}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}, D(\nabla_{\boldsymbol{x}})\boldsymbol{u})_{\boldsymbol{G}} - (\boldsymbol{u}, \boldsymbol{f})_{\boldsymbol{G}} - (\boldsymbol{u}, \boldsymbol{g})_{\boldsymbol{S}} + (\boldsymbol{A}D(\nabla_{\boldsymbol{x}})\boldsymbol{u}, \boldsymbol{h})_{\boldsymbol{T}}\,.$$

Here $(\ ,\ )_\Xi$ is the scalar product in the space $L_2(\Xi)$. Thus, the potential and elastic energies and, hence, the work of external forces are invariant with respect to affine transforms. Other invariants with respect to affine transforms of plane elastic media demanded in fracture mechanics were found in [8].

If $G = \mathbb{R}^3$ and $f^j(x) = \delta(x)e^j$, where δ is the Dirac function and e^j is the orth of the x_j axis, the solution $F^j(x)$ of system (2.1) becomes a column of the fundamental matrix $F(x)$. For an arbitrary smooth right side of f with a compact carrier, the solution u vanishing at infinity is found by the formula

$$u(y) = \int\limits_{\mathbb{R}^3} F(x-y)^{\mathrm{t}} f(x)\, dx.$$

A similar formula is valid for the corresponding solution $\boldsymbol{u}$ of system (2.6). Assuming that $y = 0$ and $\boldsymbol{y} = 0$ in these formulas and taking into account relations (2.4), we obtain the identity

$$\int\limits_{\mathbb{R}^3} F(x)^{\mathrm{t}} f(x)\, dx = u(0) = (m^{\mathrm{t}})^{-1}\boldsymbol{u}(0) = (m^{\mathrm{t}})^{-1} \int\limits_{\mathbb{R}^3} \boldsymbol{F}(\boldsymbol{x})^{\mathrm{t}}\, mf(m^{-1}\boldsymbol{x})\, d\boldsymbol{x} = \int\limits_{\mathbb{R}^3} (m^{\mathrm{t}}\boldsymbol{F}(m^{-1}x)m^{-1})^{\mathrm{t}} f(x)\, dx.$$

In the last transition, we use simple algebraic transforms and an inverse replacement of coordinates (1.4). By varying the vector function f, we now confirm relation (1.5) between the fundamental matrices $F(x)$ and $\boldsymbol{F}(\boldsymbol{x})$. The same expression relates Green's matrices for problem (2.1), (2.2) and problem (2.6), (2.7).

Thus, for anisotropic media with the matrix $\boldsymbol{A}$ of elastic moduli from the class of equivalence $\mathfrak{m}(\mathfrak{L})$, the results of [4] and simple algebraic formulas, which take into account relation (1.5), yield an explicit formula for the fundamental matrix $\boldsymbol{F}$. Unfortunately, the question about the description of the class of matrices $\mathfrak{m}(\mathfrak{L})$, which generate media algebraically equivalent to transversely isotropic ones, is still open. As was already commented in Sec. 1, this class corresponds to ten free constant.

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